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High-precision computation of the confluent hypergeometric functions via Franklin-Friedman expansion

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Abstract We present a method of high-precision computation of the confluent hypergeometric functions using an effective computational approach of what we termed *Franklin-Friedman expansions*. These expansions are convergent under mild conditions of the involved amplitude function and for some interesting cases the coefficients can be rapidly computed, thus providing a viable alternative to the conventional dichotomy between series expansion and asymptotic expansion. The present method has been extensively tested in different regimes of the parameters and compared with recently investigated convergent and uniform asymptotic expansions.

Keywords Confluent hypergeometric functions · Franklin-Friedman expansion · Uniform series expansion · Arbitrary-precision arithmetic

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1 Introduction

The current standard method of evaluation of many special functions to high-precision consists of employing the ascending series for small argument z, i.e. the direct series expansion at $z = 0$, combined with the usage of an asymptotic expansion for large values of the argument z. It is well known that the behaviour of asymptotic series of Poincaré type for large $|z|$ is characterized by having initial terms that decrease in magnitude until a minimum is attained, also known as optimal truncation, and thereafter subsequent terms start to increase. This limitation on the achievable accuracy forces a switch from the asymptotic series to the ascending series depending on the desired level of precision. However, the evaluation of ascending series for large z requires to increase the working precision

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in order to compensate the large amount of cancellation, which in turn increases the computational cost.

Normally, the scheme of computation implemented in high-precision software is a relatively simplistic choice between the ascending series and the asymptotic series based on the magnitude of the argument z and the desired level of precision or other heuristics that generally exclude other parameters involved. This type of scheme, although asymptotically valid, may lead to incorrect results in the vicinity of the transition region.

Alternative computational methods have been devised to complement the described dichotomy between series expansion and asymptotic expansion. These methods, such as *exponentially-improved* expansions [13] or their extension called *hyperasymptotic* expansions [12], are focused on extending the region of validity of the asymptotic series by iteratively re-expanding the remainder terms at optimal truncation into another asymptotic series, each exponentially smaller than its predecessor. This procedure increases the attainable accuracy of the asymptotic expansion at the expense of the computational cost of evaluating substantial complicated terms at each level of the hyperasymptotic expansion.

A remarkable method to obtain geometrically convergent series for the evaluation of special functions consists of using variants of Hadamard series, which have been extensively investigated by R. B. Paris, for example in [14]. These series involving the normalized incomplete gamma function exhibit a rapid decay after the optimal truncation term, being comparable with the behaviour of the first terms of the asymptotic expansion. A similar geometrically convergent series for the evaluation of Bessel functions was developed by D. Borwein, J. Borwein and O. Chan in [1] through the evaluation of the so-called "exp-arc" integrals.

On the other hand, it is essential to mention the role played by uniform asymptotic expansions to obtain powerful expansions valid for extended regimes of the parameters. We shall remark the so-called *vanishing saddle point* method developed by N. M. Temme in [15, 16]. This method is applicable to Laplace-type integrals of the form

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} f(t)\, dt, \tag{1}$$

with $\Re(\lambda) > 0$ and z large, in which λ may also be large. Essentially, this method expands the amplitude function $f(t)$ at $t = \mu$, $\mu = \lambda/z \geq 0$ being a uniformity parameter corresponding to the saddle point of the dominant part of the integral (1).

In this paper, we revisit the theory originally developed by J. Franklin and B. Friedman in [7], which henceforth we shall call *Franklin-Friedman* expansions. Their method was developed with the aim to overcome the disadvantages of the direct application of Watson's lemma to Laplace-type integrals [17, §2]. Historically, this method has not received significant attention, presumably due to the inherent difficulty of evaluating the coefficients of the expansion. We shall show, through the study of an important amplitude function occurring in many integral representations of special functions, how the coefficients of the Franklin-Friedman expansion can be efficiently evaluated, resulting in a convergent method capable of out-performing aforementioned methods.

The rest of the paper is outlined as follows. In Section 2, we briefly revisit the theory corresponding to the Franklin-Friedman expansion and we show an illustrative example. Then in Section 3, we compute the coefficients for the amplitude function corresponding to the confluent hypergeometric function and we provide an analysis of the obtained coefficients. In Section 4, we present an effective recursive algorithm for the computation of the coefficients and we provide numerical calculations and compare the present method with the conventional ascending-asymptotic series and previously discussed convergent and uniform asymptotic expansions. Finally, in Section 5, we discuss possible enhancements and present our conclusions.

2 The Franklin-Friedman expansion

J. Franklin and B. Friedman developed in [7] a method for obtaining convergent asymptotic representations for Laplace-type integrals of the form

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} f(t)\, dt, \quad \Re(z) > 0,\ \Re(\lambda) > 0, \tag{2}$$

for large values of z with suitable assumptions on the amplitude function $f(t)$. This method is based on the application of a type of interpolation process to the function $f(t)$, differing from Watson's lemma, in which the amplitude function is expanded in a power series at $t = 0$ and integrated term by term. The first interpolation point $t_0 = \lambda/z$ corresponds to the saddle point of the dominant part $t^\lambda e^{-zt}$, and by substituting in (2) we obtain

$$F_\lambda(z) = f(t_0) z^{-\lambda} + \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} (f(t) - f(t_0))\, dt\,. \tag{3}$$

After integrating by parts we obtain

$$F_\lambda(z) = f(t_0) z^{-\lambda} + \frac{1}{z\Gamma(\lambda)} \int_0^\infty t^{\lambda} e^{-zt} f_1(t)\, dt, \tag{4}$$

where the new amplitude function $f_1(t)$ is defined by

$$f_1(t) = \frac{d}{dt} \frac{f(t) - f(t_0)}{t - t_0}. \tag{5}$$

One can observe that integral (4) has the same form as the integral (2), with λ replaced by $\lambda + 1$ and f by f_1. The interpolation point for the next iteration is $t_1 = (\lambda + 1)/z$. This process can be continued iteratively obtaining the following series expansion

$$F_\lambda(z) = \sum_{k=0}^{n-1} f_k(t_k) \frac{(\lambda)_k}{z^{\lambda+2k}} + \frac{1}{z^n \Gamma(\lambda)} \int_0^\infty t^{\lambda+n-1} e^{-zt} f_n(t)\, dt, \tag{6}$$

where $n = 0, 1, 2, \ldots$, $f_0(t) = f(t)$ and

$$f_{k+1}(t) = \frac{d}{dt} \frac{f_k(t) - f_k(t_k)}{t - t_k}, \quad t_k = \frac{\lambda + k}{z}, \quad k = 0, 1, 2, \ldots. \tag{7}$$

Sufficient conditions on the amplitude function $f(t)$ for the convergent behaviour of the series expansion are stated in the following two theorems. We refer to [7] for proofs.

Theorem 1 (J. Franklin and B. Friedman [7]) *Take $\lambda = \alpha + i\beta$, where α is real and positive and β is real. For $\beta \neq 0$, suppose that $f(t)$ is analytic for $\Re(t) > 0$ and that $f \in C^{2n}[0, \infty)$, such that the derivatives satisfy*

$$|f^{(m)}(t)| \leq M e^{\mu t}, \quad m = 0, 1, \ldots, 2n,$$

where M and μ are non-negative constants. Under these conditions, expansion (6) has an asymptotic behaviour as $z \to \infty$, with remainder term of order $O(z^{-2n-\lambda})$. If $\lambda = \alpha$, the assumption that $f(t)$ is analytic for $\Re(t) > 0$ may be replaced by the assumption that $f \in C^{2n}[0, \infty)$ for $t > 0$. Therefore, the series expansion is convergent.

Theorem 2 (J. Franklin and B. Friedman [7]) *Suppose $f(q) = f(x + iy)$ can be represented in the form*

$$f(q) = \int_0^\infty e^{-qt}\, d\Psi(t), \quad x > 0,$$

where $\Psi(t)$ is a complex-value function which is of bounded variation in each finite interval $t \in [0, T]$ and which satisfies the inequality

$$|\Psi(t)| \leq M, \quad t \geq 0.$$

Then for $z > 0$ and $\Re(\lambda) > 0$, series expansion (6) converges to (2).

In [17, §17.4], Temme gives the first five coefficients $f_k(t_k)$ for the incomplete gamma function $e^z \Gamma(1-\lambda, z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} (1+t)^{-1}\, dt$, amplitude function $f(t) = (1+t)^{-1}$, by using computer algebra

$$\begin{aligned} f_0 &= \frac{z}{\zeta}, \quad f_1 = \frac{z^3}{\zeta(\zeta+1)^2}, \quad f_2 = \frac{z^5(3\zeta+4)}{\zeta(\zeta+1)^2(\zeta+2)^3}, \\ f_3 &= \frac{z^7(15\zeta^3 + 90\zeta^2 + 175\zeta + 108)}{\zeta(\zeta+1)^2(\zeta+2)^3(\zeta+3)^4}, \\ f_4 &= \frac{z^9(105\zeta^6 + 1680\zeta^5 + 11025\zeta^4 + 37870\zeta^3 + 71540\zeta^2 + 70120\zeta + 27648)}{\zeta(\zeta+1)^2(\zeta+2)^3(\zeta+3)^4(\zeta+4)^5}, \end{aligned} \tag{8}$$

where $\zeta = z + \lambda$. Despite the relative ease of computing the coefficients by means of computer algebra systems, it is usually difficult to obtain explicit representations such that these become usable for numerical evaluation purposes. In practice, one generates a few coefficients of the expansions for a bounded domain of the parameters and incorporate them into a routine, this procedure being solely valid for fixed precision.

3 The expansion for $U(a, b, z)$

The *confluent hypergeometric function of the first kind* ${}_1F_1(a; b; z)$ and the *Kummer function* $U(a, b, z)$ arise as linearly independent solutions of the *Kummer's differential equation* [4, §13.2]

$$z\frac{d^2w}{dz^2} + (b - z)\frac{dw}{dz} - aw = 0, \tag{9}$$

for $b \notin \mathbb{Z}^- \cup \{0\}$. Confluent hypergeometric functions appear in a wide range of applications in mathematical physics and applied mathematics. Many special functions are expressible in terms of specific forms of the confluent hypergeometric functions such as, for example, Bessel functions, incomplete Gamma functions and Laguerre polynomials amongst others.

A convenient starting point for $U(a,b,z)$ is the integral representation [17, §10.1.5]

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{-zt} (1+t)^{b-a-1}\, dt, \tag{10}$$

valid for $\Re(a) > 0$ and $\Re(z) > 0$. Laplace-type integral (10) includes the amplitude function $f(t) = (1+t)^{b-a-1}$, which we shall investigate further on. An asymptotic expansion valid for $|z| \to \infty$ can be derived by application of Watson's lemma to the integral representation (10). We obtain

$$U(a,b,z) \sim z^{-a} \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (a-b+1)_k}{k! z^k}, \quad |\mathrm{ph}\, z| \le \frac{3}{2}\pi - \delta, \tag{11}$$

where δ is an arbitrary small positive constant such that $0 < \delta \ll 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ denotes a rising factorial or Pochhammer symbol. For $\Re(z) > 0$, asymptotic series (11) is alternating and thus the remainder is bounded by the absolute value of the first neglected term. As previously discussed, the remainder cannot be reduced arbitrarily, hence when z is not sufficiently large with respect to a and b (not made rigorous here) and the required precision bits is moderate, this expansion cannot be used effectively.

Evaluation of $U(a,b,z)$ outside the sector $|\mathrm{ph}\, z| < \frac{1}{2}\pi$ can be achieved by use of the continuation formula [17, §10.1.11]

$$\begin{aligned} e^{-z} U(a,b,z) = {} & \frac{e^{\mp \pi i a} \Gamma(b-a)}{\Gamma(b)}\, {}_1F_1(b-a; b; -z) \\ & - \frac{e^{\mp \pi i b} \Gamma(b-a)}{\Gamma(a)} U(b-a, b, z e^{\mp \pi i}), \end{aligned} \tag{12}$$

where ${}_1F_1(a;b;z)$ is an entire function with series expansion given by [17, §10.1.2]

$${}_1F_1(a;b;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k k!} z^k. \tag{13}$$

To compute the Kummer function $U(a,b,z)$ for small values of z, the usual approach is to employ connection formulas for this function in terms of ${}_1F_1(a;b;z)$, for example [17, §10.1.12]

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\, {}_1F_1(a;b;z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b}\, {}_1F_1(a-b+1; 2-b; z), \tag{14}$$

which is not defined for integer values of b, although the limit exists for $b \to 0$. Additionally, a recent method for computing the Kummer function $U(a,b,z)$ for small values of $|a|$, $|b|$ and $|z|$ is described in [8].

3.1 The Franklin-Friedman expansion coefficients

We compute the coefficients of the Franklin-Friedman expansion for the amplitude function $f(t) = (1+t)^{b-a-1}$ appearing in the Laplace-type integral for $U(a,b,z)$ in (10). We start computing a few coefficients f_k in expansion (6) using Mathematica 10 [18] and employing the `Expand` and `FullSimplify` options to perform algebraic simplifications and transformations. We only show the first two coefficients since their size grows considerably with k

$$f_0 = (1 + a/z)^{b-a-1} \quad \text{and} \quad f_1 = z^3\left(\frac{(\frac{a+z}{z})^{b-a}}{a+z} - \frac{(\frac{1+a+z}{z})^{b-a}(2+2a-b+z)}{(1+a+z)^2}\right).$$

Note that above coefficients coincide with those in (8) when $\lambda = a = b$. Unfortunately, Mathematica was unable to produce more simplified expressions of f_k. Hereinafter, we proceed to generate tractable explicit representations of the coefficients f_k. Let us first define the coefficients $A_s^q := (1 + (a+s)/z)^q$. Subsequently, we factorize the previously obtained coefficients and rearrange terms such that coefficients A_j^{b-a-i} appear in ascending order (i,j). The first four coefficients f_k are now given by

$$\begin{aligned}
f_0 &= A_0^{b-a-1},\\
f_1 &= (A_1^{b-a-2}(b-a-1) + (A_0^{b-a-1} - A_1^{b-a-1})z)z,\\
f_2 &= \Big(A_2^{b-a-3}(b-a-1)(b-a-2) + 2(A_1^{b-a-2} - A_2^{b-a-2})(b-a-1)z\\
&\quad + (A_0^{b-a-1} - 2A_1^{b-a-1} + A_2^{b-a-1})z^2\Big)\frac{z^2}{2},\\
f_3 &= \Big(A_3^{b-a-4}(b-a-1)(b-a-2)(b-a-3)\\
&\quad + 3(b-a-1)(b-a-2)(A_2^{b-a-3} - A_3^{b-a-3})z\\
&\quad + 3(b-a-1)(A_1^{b-a-2} - 2A_2^{b-a-2} + A_3^{b-a-2})z^2\\
&\quad + (A_0^{b-a-1} - 3A_1^{b-a-1} + 3A_2^{b-a-1} - A_3^{b-a-1})z^3\Big)\frac{z^3}{6}.
\end{aligned}$$

We observe that the terms multiplying A_j^{b-a-i} in f_k correspond to rows of Pascal's triangle, with alternating sign for the inner terms in $(\cdots)z^{k-j}$. Furthermore, a multiplicative factor $z^k/k!$ is present. After performing a few more algebraic manipulations we obtain the explicit representation of f_k given by

$$f_k = c_k(z)\frac{z^k}{k!}, \tag{15}$$

where

$$c_k(z) = \sum_{j=0}^{k}\binom{k}{j}z^{k-j}d_j\sum_{s=j}^{k}(-1)^{s-j}\binom{k-j}{k-s}A_s^{b-a-1-j}, \quad k = 0,1,2,\ldots \tag{16}$$

and d_j is defined by

$$d_j = \frac{\Gamma(b-a)}{\Gamma(b-a-j)} = \begin{cases}1, & j = 0\\ d_{j-1}(b-a-j), & j > 0\end{cases} \tag{17}$$

We remark that equations (16)-(17) add the internal coefficients backwards. In order to calculate them forward, the coefficients $c_k(z)$ are written equivalently as

$$c_k(z) = \sum_{j=0}^{k} \binom{k}{j} z^j l_j \sum_{s=0}^{j} (-1)^s \binom{j}{s} A_{k-j+s}^{b-a-1-k+j}, \tag{18}$$

where

$$l_j = \frac{\Gamma(b-a)}{\Gamma(b-a+j-k)} = \begin{cases} \prod_{i=1}^{k}(b-a-i), & j=0 \\ \frac{d_{j-1}}{(b-a-k+j)}, & j>0 \end{cases} \tag{19}$$

Note that the previous double finite summation is over the triangle $0 \le s \le j \le k$. Furthermore, $c_k(z)$ can be written as

$$c_k(z) = z^{k-q} k! \sum_{j=0}^{k} \binom{q}{k-j} \frac{1}{j!} \sum_{s=0}^{j} (-1)^s \binom{j}{s} \frac{1}{(p+k-j+s)^{k-q-j}}, \tag{20}$$

where $q = b-a-1$ and $p = z+a$. We shall see that the explicit representation in (20) will lead to our main result (Theorem 3), where we prove that the following expression for $U(a,b,z)$ holds

$$U(a,b,z) = \sum_{k=0}^{\infty} c_k(z) \frac{(a)_k}{k! z^{a+k}}. \tag{21}$$

3.2 Analysis of the coefficients $c_k(z)$

In this subsection we examine the coefficients $c_k(z)$ defined in (20). Let us define the coefficients $r_j := \sum_{s=0}^{j} (-1)^s \binom{j}{s} (p+k-j+s)^{q+j-k}$ corresponding to the inner summation in (20). These coefficients are defined by the binomial transform of the sequence $\{(p+k-j+s)^{q+j-k}\}_{s\ge 0}$, which can be represented by means of the Nörlund-Rice integrals [6].

In what follows, we proceed to derive asymptotic expansions and upper bounds for the coefficients $c_k(z)$. Let us consider the alternating binomial sum defined by

$$F(z,N,m) = \sum_{k=0}^{N} \binom{N}{k} (-1)^k \frac{1}{(z+k)^m}. \tag{22}$$

In [3], Coffey calculates the alternating binomial sum (22) for $(N,m) \in \mathbb{N}$ and $z \in \mathbb{C} \backslash \mathbb{Z}_0^-$. The main result of [3] is an analytic relation in terms of Bell polynomials with generalized harmonic number arguments. We present some of his results along with other relations that will be needed further on

$$F(z,N,m) = \frac{1}{z^m} {}_{m+1}F_m(z,\ldots,z,-N; z+1,\ldots,z+1; 1) \tag{23}$$

$$= \frac{1}{(m-1)!} \int_0^\infty t^{m-1} e^{-zt} (1-e^{-t})^N \, dt \tag{24}$$

where ${}_pF_q(a_1,\ldots,a_p; b_1,\ldots,b_q; z)$ is the generalized hypergeometric function.

The Stirling numbers of the second kind $S(n,k)$ may be defined by the following generating function

$$\sum_{k=1}^{n} S(n,k)(x-k+1)_k = x^n, \tag{25}$$

and $S(n,k) = 0$ for $n < k$. We introduce the following proposition for the coefficients $c_k(z)$.

Proposition 1 *Given $a, b, z \in \mathbb{C}$, $\Re(z) > 0$ and $j, k \in \mathbb{N}$, coefficients $c_k(z)$ in (20) can be represented by the following asymptotic expansion as $(p+k) \to \infty$*

$$c_k(z) \sim z^{k-q} k! \sum_{j=0}^{k} \binom{q}{k-j} (j+1) \sum_{i=j+1}^{\infty} \frac{S(i,j+1)(k-q-j)_i}{i!} \zeta(k-q-j+i, p+k+1). \tag{26}$$

Proof We use the integral representation (24) for r_j

$$r_j = \frac{1}{\Gamma(k-q-j)} \int_0^\infty t^{k-q-j-1} e^{-(p+k-j)t} (1-e^{-t})^j \, dt \,. \tag{27}$$

This Laplace-type integral is defined for $\Re(k-q-j) > 0$, where a direct use of Watson's lemma applies, see (30). Alternatively, note that this integral can be written as

$$\int_0^\infty t^{k-q-j-1} e^{-(p+k-j)t} (1-e^{-t})^j \, dt = \int_0^\infty t^{k-q-j-1} e^{-(p+k+1)t} \frac{(e^t-1)^{j+1}}{(1-e^{-t})} \, dt \,.$$

An asymptotic expansion for r_j can be obtained if we expand $(e^t-1)^{j+1}$ at $t=0$, which corresponds to the following generating function of the Stirling coefficients of the second kind (25)

$$(e^t-1)^{j+1} = (j+1)! \sum_{i=j+1}^{\infty} S(i,j+1) \frac{t^i}{i!}, \tag{28}$$

and interchanging the previous summation and integration we obtain a divergent asymptotic series

$$r_j \sim \frac{(j+1)!}{\Gamma(k-q-j)} \sum_{i=j+1}^{\infty} \frac{S(i,j+1)}{i!} \int_0^\infty t^{k-q-j+i-1} \frac{e^{-(p+k+1)t}}{(1-e^{-t})} \, dt \,. \tag{29}$$

The resulting integral has an explicit representation in terms of the Hurwitz zeta function [4, §25.11]. Hence, replacing the integral by the Hurwitz zeta function and substituting the ratio of gamma functions by a Pochhammer symbol gives the result. □

An equivalent asymptotic expansion representation for $(p+k) \to \infty$ is obtained by application of Watson's lemma to integral (27)

$$r_j \sim \frac{j!}{\Gamma(k-q-j)} \sum_{i=j}^{\infty} \frac{S(i,j)}{i!} \frac{\Gamma(k-q-j+i)}{(p+k)^{i+k-q-1}} = j! \sum_{i=j}^{\infty} \frac{S(i,j)}{i!} \frac{(k-q-j)_i}{(p+k)^{i+k-q-1}}, \tag{30}$$

where the use of Pochhammer symbol permits the evaluation out of the domain $\Re(k-q-j) > 0$. In [3] a similar asymptotic expansion restricted to $k-q-j \in \mathbb{N}$ is obtained after a change of variable and using the generating function for Stirling numbers of the first kind and solving the beta function.

Remark 1 Interestingly, coefficients $c_k(z)$ have a remarkable property for (a,b) in a domain $\mathcal{D} := \{(a,b) \in \mathbb{C}^2 : b-a-1 = n, n \in \mathbb{N}\}$. For this particular case, $U(a,b,z)$ reduces to a polynomial in z of degree n given by

$$U(a, a+n+1, z) = z^{-a} \sum_{j=0}^{n} \binom{n}{j} \frac{(a)_j}{z^j}. \tag{31}$$

We note, by expanding the double binomial sum in (20), that coefficients $c_k(z)$ vanish by symmetry for $k > \lfloor n/2 \rfloor$. This implies that expansion (21) terminates in almost half of terms required by (31).

We now prove the Franklin-Friedman expansion for $U(a,b,z)$ in (21).

Theorem 3 *For $(a,b,z) \in \mathbb{C}^3$ and $\Re(z) > 0$ the Franklin-Friedman expansion for the confluent hypergeometric function $U(a,b,z)$ is given by*

$$U(a,b,z) = \sum_{k=0}^{\infty} c_k(z) \frac{(a)_k}{k! z^{a+k}}, \tag{32}$$

where

$$c_k(z) = \sum_{j=0}^{k} \binom{k}{j} z^{k-j} \frac{\Gamma(b-a)}{\Gamma(b-a-j)} \sum_{s=j}^{k} (-1)^{s-j} \binom{k-j}{k-s} A_s^{b-a-1-j} \tag{33}$$

$$= \sum_{j=0}^{k} \binom{k}{j} z^{j} \frac{\Gamma(b-a)}{\Gamma(b-a+j-k)} \sum_{s=0}^{j} (-1)^{s} \binom{j}{s} A_{k-j+s}^{b-a-1-k+j}, \quad k = 0,1,2,\ldots, \tag{34}$$

and

$$A_s^q = \left(1 + \frac{a+s}{z}\right)^q. \tag{35}$$

Proof It follows from (24) that coefficients $c_k(z)$ can be written as

$$\begin{aligned} c_k(z) &= \frac{k!}{z^{q-k}} \sum_{j=0}^{k} \binom{q}{k-j} \frac{1}{j!} \sum_{s=0}^{j} (-1)^s \binom{j}{s} \frac{1}{(p+k-j+s)^{k-q-j}} \\ &= \frac{k!}{z^{q-k}} \sum_{j=0}^{k} \binom{q}{k-j} \frac{1}{j! \Gamma(k-j-q)} \int_0^{\infty} t^{k-q-j-1} e^{-(p+k-j)t} (1-e^{-t})^j \, dt, \end{aligned} \tag{36}$$

where $q = b-a-1$ and $p = z+a$ with the integral representation being valid for $\Re(k-q-j) > 0$ and $\Re(p+k-j) > 0$. Next, we rearrange $U := \binom{q}{k-j} \frac{1}{j!\Gamma(k-j-q)}$, to obtain $U = u(q,k,j) f(k,j)$, where $f(k,j) = 1/(j!(k-j)!)$. The following identity is established for $u(q,k,j)$

$$u(q,k,j) = \frac{\Gamma(q+1)}{\Gamma(q-k+j+1)\Gamma(k-j-q)} = -\frac{\Gamma(q+1)\sin(\pi(j-k+q))}{\pi}. \tag{37}$$

Now replacing (37) into (36) and by formally interchanging integration and summation we obtain the following integral representation for $c_k(z)$

$$
\begin{aligned}
c_k(z) &= -\frac{\Gamma(q+1)k!}{z^{q-k}\pi}\int_0^\infty \left(\sum_{j=0}^{k}\frac{\sin(\pi(j-k+q))e^{jt}(1-e^{-t})^j}{j!(k-j)!t^j}\right)t^{k-q-1}e^{-(p+k)t}\,dt \\
&= \frac{\Gamma(q+1)\sin(\pi(k-q))}{z^{q-k}\pi}\int_0^\infty t^{-q-1}e^{-(p+k)t}(1-e^t+t)^k\,dt, \qquad (38)
\end{aligned}
$$

valid for $\Re(q) < 0$ and $\Re(p) > 0$. Given the explicit integral representation of $c_k(z)$ in (38), the proof consists of developing the sum after replacing (38) into (32). Thus, we obtain after interchanging summation and integration

$$
\sum_{k=0}^{\infty}\frac{c_k(z)(a)_k}{k!z^{a+k}} = \frac{\Gamma(q+1)}{\pi z^{a+q}}\int_0^\infty\left(\sum_{k=0}^{\infty}\frac{(a)_k\sin(\pi(k-q))(1-e^t+t)^k}{e^{kt}k!}\right)t^{-q-1}e^{-pt}\,dt\,.
$$

This interchange is justified since the series converges absolutely. Now we consider the following identity

$$
\sum_{k=0}^{\infty}\frac{(a)_k\sin(\pi(k-q))(1-e^t+t)^k}{e^{kt}k!} = -\frac{\sin(\pi q)e^{at}}{(1+t)^a}. \qquad (39)
$$

Using (39), it follows that the Franklin-Friedman expansion in (32) is expressible in terms of the Laplace-type integral given by

$$
\begin{aligned}
\sum_{k=0}^{\infty}c_k(z)\frac{(a)_k}{k!z^{a+k}} &= -\frac{\Gamma(q+1)\sin(\pi q)}{\pi z^{a+q}}\int_0^\infty\frac{t^{-q-1}e^{-(p-a)t}}{(1+t)^a}\,dt \\
&= -\frac{\Gamma(b-a)\sin(\pi(b-a-1))}{\pi z^{b-1}}\int_0^\infty\frac{t^{a-b}e^{-zt}}{(1+t)^a}\,dt\,.
\end{aligned}
$$

By observing that the resulting integral is indeed the integral representation of $U(a,b,z)$ in (10) after application of Kummer's transformation $U(a,b,z) = z^{1-b}U(1+a-b,2-b,z)$, we obtain

$$
\begin{aligned}
U(a,b,z) &= -\frac{\Gamma(b-a)\sin(\pi(b-a-1))z^{1-b}}{\pi}\int_0^\infty\frac{t^{a-b}e^{-zt}}{(1+t)^a}\,dt \qquad (40)\\
&= \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}e^{-zt}(1+t)^{b-a-1}\,dt, \qquad (41)
\end{aligned}
$$

and the proof of the theorem is completed. □

Remark 2 A relation between contiguous coefficients can be obtained by performing integration by parts on (38), which yields

$$
c_{k+1}(p,q;z) = z(c_k(p,q;z) - c_k(p+1,q;z)) + qc_k(p+1,q-1;z). \qquad (42)
$$

Despite the interest of the latter result, this recurrence is not the preferred choice to compute consecutive coefficients. An efficient method to compute a set of coefficients $c_k(z)$ is described in Section 4.

To conclude the analysis of coefficients $c_k(z)$, we derive an upper bound for the domain of parameters $\mathcal{D} := \{(a, b, z) \in \mathbb{C}^3 : \Re(a) > \Re(b) - 1 \wedge \Re(z + a) > 0\}$.

Proposition 2 *For $\Re(q) < 0$ and $\Re(p) > 0$, we have*

$$|c_k(z)| \leq |z^{k-q} p^q|. \tag{43}$$

Proof Let us consider the integral representation of $c_k(z)$ in (38)

$$c_k(z) = \varphi(k, p, q) \int_0^\infty t^{-q-1} e^{-(p+k)t} (1 - e^t + t)^k \, dt,$$

where

$$\varphi(k, p, q) = \frac{z^{k-q} \Gamma(q+1) \sin(\pi(k-q))}{\pi} = (-1)^k \frac{z^{k-q}}{\Gamma(-q)}.$$

We have that the amplitude function $(1 - e^t + t)^k$ is bounded in absolute value by $|1 - e^t + t|^k \leq (1+t)^k + e^{kt}$. Hence,

$$|c_k(z)| \leq |\varphi(k, p, q)| \left(\int_0^\infty t^{-q-1} e^{-(p+k)t} (1+t)^k \, dt + \int_0^\infty t^{-q-1} e^{-pt} \, dt \right). \tag{44}$$

We note that the first integral is expressible in terms of Charlier polynomials of degree q [4, §13.6.20]

$$\frac{1}{\Gamma(-q)} \int_0^\infty t^{-q-1} e^{-(p+k)t} (1+t)^k \, dt = (k+p)^q C_q(k; k+p), \tag{45}$$

whereas the second integral is simply

$$\int_0^\infty t^{-q-1} e^{-pt} \, dt = \Gamma(-q) p^q. \tag{46}$$

Substituting (45) and (46) into (44) gives an upper bound for $c_k(z)$. A simpler bound can be easily obtained by means of the equivalent integral representation

$$c_k(z) = \frac{z^{k-q}}{\Gamma(-q)} \int_0^\infty t^{-q-1} e^{-pt} (1 - (1+t)e^{-t})^k \, dt,$$

where the amplitude function is bounded in absolute value by $|1 - (1+t)e^{-t}| \leq 1$. Thus, using the result in (46) gives the bound. □

Table 1 shows effectiveness of the bound for several values $(a, b, z) \in \mathcal{D}$ and k. Such a bound can be used to determine the truncation level N due to the convergent behaviour of the expansion.

a	b	z	k	$\lvert c_k \rvert$	(43)
-500.1	-602.4	770	10	1.1e+64	7.9e+75
710.2	72.5	1500	15	2.7e−82	1.3e−60
-50.1	-62.4	70	20	2.8e+34	1.5e+44
-10.1	-52.4	80	30	1.3e+40	4.3e+59
-5.1	-62.4	40	50	5.4e+73	3.6e+83

Table 1: Effectiveness of bound on $c_k(z)$ in (43) for $q < 0$ and $p > 0$.

Furthermore, application of Watson's lemma to integral representation (38) gives a first-order asymptotic approximation for $(p+k) \to \infty$ given by

$$c_k(z) \sim \frac{z^{k-q}\Gamma(q+1)\sin(\pi(k-q))}{\pi}\frac{\Gamma(2k-q)}{2^k(k+p)^{2k-q}} = (-1)^k\frac{z^{k-q}(-q)_{2k}}{2^k(k+p)^{2k-q}}. \quad (47)$$

Table 2 shows a few examples of the above asymptotic approximation of $c_k(z)$ for large argument.

a	b	z	k	$c_k(z)$	(47)
12.1	10.4	100	10	5.7e−05	4.6e−05
22.1	11.4	100	50	1.7e+36	1.8e+38
20.1	12.4	280	50	3.9e+23	3.6e+25

Table 2: Asymptotic approximation on $c_k(z)$ in (47) for $q < 0$ as $p + k \to \infty$.

Finally, we note that the terms of the Franklin-Friedman expansion (21) satisfy the order estimate

$$c_k(z)\frac{(a)_k}{k!z^{k+a}} = O(2^k k^{a-3/2} e^{-2k} h(z)), \quad k \to \infty, \quad p \to \infty, \quad (48)$$

where $h(z) = (1 + p/k)^{-2k+q}$. This result is obtained combining the asymptotic estimate in (47) and the usual estimates for the factorial and Pochhammer symbol given by

$$k! = O\left(k^{k+1/2}e^{-k}\right), \quad (a)_k = O\left(k^{a+k-1/2}e^{-k}\right), \quad k \to \infty.$$

4 Efficient computation of $U(a, b, z)$

4.1 Fast computation of coefficients $c_k(z)$

Equations (16), (18) and (20) are explicit representations that can be used to compute $c_k(z)$ directly, but a double binomial sum turns out to be significantly expensive as k grows. Furthermore, from a numerical perspective, the evaluation of alternating binomial sums are prone to suffer from substantial cancellation. As we shall see, the direct computation of $c_k(z)$ can be avoided by constructing a recurrence equation for generating a set of $c_k(z)$, $k \in \{0, \ldots, N\}$. This idea leads to the following proposition and Algorithm 1.

Proposition 3 *The coefficients $c_k(z)$ of the Franklin-Friedman expansion for the amplitude function $f(t) = (1+t)^{b-a-1}$ satisfy the recurrence equation*

$$c_k(z) = u_k + \sum_{i=0}^{k-1}\binom{k}{i} z^{k-i} u_i, \quad (49)$$

where

$$u_k = A_k^{b-a-1-k} k! L_k^{b-a-1-k}(z+a+k) \quad \text{and} \quad c_0 = u_0 = A_0^{b-a-1}, \quad (50)$$

$L_k^{\lambda}(z)$ being generalized Laguerre polynomials.

Proof We start defining u_k as the leading coefficient $f(A_k)$, resulting from the iteration $j = 0$ in (18). After grouping the remaining coefficients $f(A_j)$, $j < k$ in ascending order j, we identify that each $f(A_j)$ is a non-linear combination of the previous leading terms u_j, $j < k$. These non-linear terms in the recurrence equation are indeed binomial coefficients $\binom{k}{j}$ times z^{k-j}. This gives the following expression for u_k, which can be expressed in terms of generalized Laguerre polynomials as follows

$$\begin{aligned} u_k &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} z^{k-j} d_j A_k^{b-a-1-j} \\ &= A_k^{b-a-1-k} \sum_{j=0}^{k} \binom{k}{j} (-zA_k)^{k-j} \frac{\Gamma(b-a)}{\Gamma(b-a-j)} \\ &= A_k^{b-a-1-k} k! L_k^{b-a-1-k}(zA_k) \end{aligned} \tag{51}$$

and d_j is defined as in (17). Taking $zA_k = z + a + k$ gives the final closed form for u_k. □

Recently, several variants of asymptotic expansions for large order k have been extensively studied for generalized Laguerre polynomials. For example, the paper [2] provides a treatment for the region of sub-exponential behaviour and the recent paper [5] studies uniform asymptotic expansions for a larger domain of the parameters. Although the computation of $c_k(z)$ by means of computing Laguerre polynomials would certainly reduce cancellation effects, there is a substantial computational cost involved.

In order to bypass the computation of generalized Laguerre polynomials and the direct computation in (20), we introduce a fast algorithm for computing $c_k(z)$, see Algorithm 1. This algorithm combines both binomial expansion sums to reuse the binomial coefficients, so in practice $c_k(z)$ at u_k are computed at once. In terms of time complexity, computing the first N coefficients $c_k(z)$ using Algorithm 1 has complexity $O(N^2)$, whereas clearly a direct computation has complexity $O(N^3)$. Additionally, note that coefficients d_j in (17) do not need to be computed for each j but for k, thus avoiding redundant operations. In terms of algorithmic aspects, given a suitable truncation level N, the complete Pascal's triangle until row N can be pre-computed with complexity $O(N^2)$, obviously computing only half rows. On the other hand, in terms of space complexity, computing the first N coefficients with Algorithm 1 has complexity $O(2N)$, due to the storage of successive u_k and d_k. As we shall see later, this is not an issue due to the small number of terms needed to obtain high accuracy. In general, several parts of the algorithm can be easily pre-computed in parallel, for example a block of $k : k \leq N$ coefficients $A_k^{b-a-1-j}$ can be distributed to each thread. However, although pre-computations improve efficiency, they require substantial space, especially for high-precision computations. Concerning working precision, higher precision arithmetic is normally needed to satisfy the requested accuracy, especially for large values of the parameters and argument. Based on experiments, we found that to guarantee good performance a working precision of p bits must satisfy $p \gtrsim 2N$, N being the number of terms in expansion (21). For this same reason, approaches based on performing linear search to obtain an optimal number of terms N using floating-point arithmetic are less likely to succeed.

Algorithm 1 Fast computation of coefficients $c_k(z)$

Input: $a, b, z \in \mathbb{C}$, $\Re(z) > 0$, $N \in \mathbb{N}$
Output: $c_k(z)$, $k \in \{0, \ldots, N\}$

```
Pre-compute Pascal's triangle, PascalRow(k), k ∈ {0,...,N}
U = [ ], C = [ ], D = [ ]                          ▷ Empty cache of coefficients c, u and d
U[0] ← C[0] ← (1 + a/z)^(b−a−1) and D[0,1] ← [1, b − a − 1]
for k = 1; k = N; k ← k + 1 do
    r ← 0, t ← 0
    f ← z^k
    h ← 1/z
    m ← 1 + (a + k)/z
    n ← 1/m
    q ← m^(b−a−1)
    R = PascalRow(k)                               ▷ k-th row of pre-computed Pascal's triangle
    for j = 0; j = k; j ← j + 1 do
        u ← f · R[j]
        p ← (−1)^(k−j) · u · D[j] · q
        t ← t + p
        f ← f · h
        q ← q · n
        if j < k then
            r ← r + U[j] · u
        else
            D[j + 1] ← D[j] · (b − a − 1 − j)      ▷ Store next d
        end if
    end for
    U[k] ← t                                       ▷ Store block u_k to posterior use
    r ← r + t
    C[k] ← r                                       ▷ Store new coefficient c_k(z)
end for
```

4.2 Numerical experiments

In this Section we compare expansion (21) with other asymptotic and convergent series previously mentioned. The described algorithms has been implemented in Python using the Mpmath library for multi-precision floating-point arithmetic [10]. Firstly, we compare (21) with the convergent expansion for $U(a, b, z)$ in [14], which we briefly summarize:

$$U(a, a+b, z) = z^{-a} \sum_{k=0}^{\infty} \frac{(-1)^k (a)_k (1-b)_k}{k! z^k} P(k+a, \vartheta x) + \frac{e^{-ia\theta}}{\Gamma(a)} \sum_{n=1}^{\infty} e^{-\Omega_n x} S_n(x;\theta), \tag{52}$$

valid in $|\arg z| < \pi$ with $z = xe^{i\theta}$ and $x = |z|$. The second series is defined as

$$S_n(x;\theta) = \sum_{k=0}^{\infty} \frac{c_{k,n}(\theta)}{x^{k+1}} \Delta P\left(k+1, \frac{\omega_n x}{2}\right), \quad \Delta P(m+1, z) = P(m+1, z) - P(m+1, -z).$$

Coefficients $c_{k,n}(\theta)$ can be computed via recurrence relation. This method subdivides the integration path in (10) into intervals of length $\frac{1}{2}\omega_n$, where $\frac{1}{2}\omega_n = \vartheta$, $\vartheta \in (0, 1]$ and Ω_n denote the mid-points of these intervals, being both freely chosen. As shown in [14, §4], expansion (52) converges geometrically without restriction on the parameters a and b when $\vartheta < 1$. It is observed that evaluation of the second series may be unavoidably expensive, even though the incomplete gamma function can be computed via recursion.

We consider the example in [14, §5] to compare both expansions. This example evaluates the modified Bessel function $K_\nu(z)$ given by

$$e^z K_\nu(z) = \sqrt{\pi}(2z)^\nu U\left(\nu + \frac{1}{2}, 2\nu + 1, 2z\right). \tag{53}$$

We reproduce Table 3 in [14, §5] for different values of θ with convergence rate $\vartheta = \frac{1}{2}$ and $n = 2$, i.e. using the first two terms of the second expansion in (52). We skip the improved case $\vartheta = \frac{1}{3}$ and $n = 4$ in [14, §5] due to the notable computational effort. Results reveal that for $z \in \mathbb{R}$, expansion (21) gives more correct digits than Hadamard expansion, but for $\Re(z) \leq \Im(z)$ expansion (21) is affected by a progressive loss of accuracy due to the omission of path rotation arguments like those employed in [14].

(θ/π)	**Hadamard (52)**	**Expansion (21)**
0	4.3e−43	1.3e−50
0.125	8.3e−43	2.4e−49
0.250	9.8e−43	3.0e−45
0.375	5.3e−43	1.3e−38
0.500	6.4e−43	2.3e−28

Table 3: Comparison of the absolute error values when $z = 15e^{i\theta}$ and $\nu = \frac{3}{4}$. Series truncated at $N = 100$ terms.

In the following tables we compare the performance of the ascending series, asymptotic series and the vanishing saddle point series. The vanishing saddle point series for $U(a, b, z)$ is given by [17, §25.4]

$$U(a, b, z) \sim \sum_{k=0}^{\infty} \frac{(1+\mu)^{b-a-1-k}\binom{b-a-1}{k} P_k(a)}{z^{a+k}}, \tag{54}$$

where $\mu = a/z$ is the saddle point of the dominant part of the integral (10). Coefficients $P_k(a)$ are expressible in terms of generalized Laguerre polynomials defined by

$$P_k(a) = k! L_k^{-n-a}(-a), \tag{55}$$

and satisfy the following recursion relation

$$P_0(a) = 1, \quad P_1(a) = 0, \quad \text{and} \quad P_{k+1}(a) = k(P_k(a) + aP_{k-1}(a)), \quad k = 1, 2, \ldots. \tag{56}$$

For comparison, absolute relative errors are computed using as a reference Arb [9] `hypergeometric_U` evaluated at 5000-10000 bits of precision. Symbol (-) indicates an absolute error > 1. Table 4 shows the absolute relative errors for large parameters and argument when truncated at N terms. For these cases each term of expansion (21) adds about 1 digit of accuracy every term. In particular, the first case in Table 4 corresponds to the generalized exponential integral $E_\nu(z) = z^{\nu-1} U(\nu, \nu, z)$. We remark that this special case can be computed with

(a,b,z)	N	Asymptotic (11)	Vanishing (54)	Expansion (21)
	10	-	3e−14	6e−25
	30	-	7e−34	1e−59
$(600,600,500)$	50	-	7e−50	6e−88
	100	-	4e−82	5e−146
	200	-	3e−128	2e−234
	10	-	1e−04	1e−10
	30	-	4e−15	8e−38
$(100,1,1000)$	50	4e−02	2e−26	7e−66
	100	1e−27	2e−54	4e−132
	200	9e−60	1e−102	2e−245
	10	-	3e−01	3e−03
	30	-	2e−05	2e−17
$(1000,500,5000)$	50	-	1e−11	2e−36
	100	-	2e−31	8e−92
	200	-	4e−78	3e−213

Table 4: Comparison between various methods for $U(a,b,z)$. Large parameters and argument.

faster methods, for example the Laguerre expansion described in [11] returns an absolute relative error of magnitude 1.4e−294 with $N=200$ terms.

Table 5 shows the performance of expansion (21) for small and moderate positive values of parameters and argument. For theses cases, we incorporate the ascending series in (14) into the first column, where the value within parentheses indicates the required number of terms to obtain that absolute relative error. We notice that for sufficiently small argument z the ascending series out-performs the Franklin-Friedman expansion, although for moderate values the latter substantially improves both the ascending and asymptotic expansion.

(a,b,z)	N	Ascending (14)	Asymptotic (11)	Vanishing (54)	Expansion (21)
	10	-	1e−04	6e−10	8e−20
	30	-	1e−16	8e−26	1e−52
$(30,81/4,300)$	50	-	1e−27	8e−39	3e−79
	100	-	5e−48	6e−63	5e−131
	200	1e−13 ($N=1000$)	4e−68	3e−88	4e−202
	10	-	-	8e−04	6e−08
	30	-	-	3e−07	7e−20
$(123/4,101/5,50)$	50	-	-	7e−08	2e−28
	100	-	-	-	4e−43
	300	2e−48	-	-	9e−71
	10	-	1e−10	2e−11	8e−19
	30	-	5e−15	4e−16	4e−32
$(5/4,10/4,30)$	50	-	7e−13	3e−14	2e−39
	100	6e−11	-	-	4e−50
	200	6e−80	-	-	4e−61
	10	-	-	-	7e−01
	30	-	-	-	1e−04
$(401/2,211/6,300)$	50	-	-	-	5e−11
	100	-	-	-	6e−30
	200	4e−53 ($N=1600$)	-	-	3e−66

Table 5: Comparison between various methods for $U(a,b,z)$. Small and moderate values of parameters and argument.

Finally, Table 6 shows cases with moderate negative parameters and positive argument. For theses cases expansion (21) exhibits fast convergence giving about 1.7 digits of accuracy every term. We remark that, as described in [14], Hadamard series accuracy deteriorates when a is large and negative due to the oscillatory behaviour of the incomplete gamma function.

(a,b,z)	N	Asymptotic (11)	Vanishing (54)	Expansion (21)
	10	-	-	-
	30	-	-	-
$(-241/2, 20, 400)$	50	-	-	2e−19
	100	1e−11	3e−16	2e−170
	200	1e−226	4e−138	2e−375
	10	-	-	-
	30	-	-	3e−21
$(-500/6, -21/6, 300)$	50	-	9e−11	2e−87
	100	9e−118	7e−78	9e−212
	200	6e−225	1e−143	3e−334

Table 6: Comparison between various methods for $U(a,b,z)$. Moderate negative parameters and argument.

5 Discussion

The Franklin-Friedman expansion developed provides a uniform approach to evaluating confluent hypergeometric functions to arbitrary-precision for sufficiently large $\Re(z) > 0$ and outside this sector via connection formulas. This expansion is generally convergent and especially useful for the transition region between the ascending and asymptotic series. It is found that the expansion developed systematically out-performs the vanishing saddle point series and asymptotic expansion in their respective regions of validity and is remarkably useful to compute the confluent hypergeometric function for large parameters and argument, providing a clear advantage over direct computation using the ascending series. Furthermore, it is observed that for small argument the expansion developed converges at approximately geometric rate 2^{-N}. However, for sufficiently small argument the ascending series still exhibits faster convergence.

The presented approach results adequate in a "medium/high-precision" range, say 100 - 1000 digits, given the considerable computation complexity as the number of terms increases. Therefore, a complete arbitrary-precision implementation should combine the Franklin-Friedman expansion with other low-complexity methods described in this work.

Finally, further work is needed to compute uniform bounds for the coefficients of the expansion in a wider regions of the parameters a and b, and to enhance the algorithm to accelerate the computation of coefficients via efficient parallelization.

References

1. D. Borwein, J. M. Borwein, and O. Chan. The evaluation of Bessel functions via exp-arc integrals. *Journal of Mathematical Analysis and Applications*, 341(1):478–500, 2008.
2. D. Borwein, J. M. Borwein, and R. E. Crandall. Effective Laguerre asymptotics. *SIAM J. Numerical Analysis*, 46(6):3285–3312, 2008.
3. W. M. Coffey. A set of identities for a class of alternating binomial sums arising in computing applications. *Util. Math.*, 2007.
4. *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov/, Release 1.0.14 of 2016-12-21. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.
5. T.M. Dunster, A. Gil, and J. Segura. Uniform asymptotic expansions for Laguerre polynomials and related confluent hypergeometric functions. 2017.
6. P. Flajolet and R. Sedgewick. Mellin transforms and asymptotics: Finite differences and Rice's integrals. *Theoretical Computer Science*, (144):101–124, 1995.

7. J. Franklin and B. Friedman. A convergent asymptotic representation for integrals. *Mathematical Proceedings of the Cambridge Philosophical Society*, 53(3):612–619, 1957.
8. A. Gil, J. Segura, and N. M. Temme. Computing the Kummer function $U(a, b, z)$ for small values of the arguments. *Applied Mathematics and Computation*, 271:532 – 539, 2015.
9. F. Johansson. Arb: a C library for ball arithmetic. *ACM Communications in Computer Algebra*, 47(4):166–169, 2013.
10. F. Johansson et al. *mpmath: a Python library for arbitrary-precision floating-point arithmetic (version 0.19)*, December 2014. `http://mpmath.org/`.
11. G. Navas-Palencia. Fast and accurate algorithm for the generalized exponential integral $E_\nu(x)$ for positive real order. *Numerical Algorithms*, pages 1–28, 2017.
12. A. B. Olde Daalhuis and F. W. J. Olver. Hyperasymptotic solutions of second-order linear differential equations. I. *Methods Appl. Anal.*, 2(2):173197, 1995.
13. F. W. J. Olver. Uniform, exponentially improved, asymptotic expansions for the confluent hypergeometric function and other integral transforms. *SIAM J. Math. Anal.*, 22(5):14751489, 1991.
14. R.B. Paris. High-precision evaluation of the Bessel functions via Hadamard series. *Journal of Computational and Applied Mathematics*, 224(1):84 – 100, 2009.
15. N. M. Temme. Uniform asymptotic expansions for Laplace integrals. *Analysis*, 3:221–249, 1983.
16. N. M. Temme. Laplace type integrals: Transformations to standard form and uniform asymptotic expansions. *Quarterly of Applied Mathematics*, 43(1):103–123, 1985.
17. N. M. Temme. *Asymptotic Methods for Integrals*, volume 6 of *Series in Analysis*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.
18. Wolfram Research, Inc. Mathematica 10.